Extremal Problems Concerning Transformations of the Edges of the Complete Hypergraphs

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ABSTRACT

We consider extremal problems concerning transformations of the edges of complete hypergraphs. We estimate the order of the largest subhypergraph K such that for every edge $e \in E(K)$, $f(e) \notin E(K)$, assuming $f(e) \neq e$. Several extensions and variations of this problem are also discussed here.

INTRODUCTION

In this paper we investigate some extremal problems concerning transformations of the n-subsets of a set. We begin with some technical definitions.

For a set S, let $|S|$ denote its cardinality. For a graph G, let $V(G)$ [respectively, $E(G)$] denote the vertex set (respectively, edge set) of G, and let $e(G) = |E(G)|$. Use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degree of the vertices of G, respectively. For $u \in V(G)$, let deg u denote its degree. For the complete graph on m vertices K_m, we sometimes use E_m for $E(K_m)$. Let $|S|^n$ denote the set of all n-subsets of S, and if $|S| = p$, write K_p^n for $[S]^n$. The complete q-partite graph with all color classes of size p is denoted by $K_q(p)$. Finally, Aut G denotes the automorphism group of the graph G.

Let $f: [S]^n \to [S]$. The *order* of f is

$$d(f) = \max_{v \in [S]^n} |v \cap f(v)|.$$

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Let $F(m,n,d) = \{f:[S]^n \to [S]^n \mid d(f) = d,\ |S| = m\}$. For $A \subseteq [S]^n$, and $f:[S]^n \to [S]^n$, say that A is *f-free* if for all $a \in A$, $f(a) \notin A$. Define

$$g(p,n,d) = \max\{m \mid \text{there is } f \in F(m,n,d) \text{ with no copy of } K_p^n\ f\text{-free}\}.$$

(Note that $g(p,n,d)$ is well defined for $p \geq 2n - d$.) Let

$$g(G,d) = \max\{m \mid \text{there is } f \in F(m,2,d) \text{ with no copy of } G\ f\text{-free}\},$$

and only consider cases where this exists.

Define

$$H(m,n,d) = \{f:E(K_n(m)) \to E(K_n(m)) \mid d(f) = d \in \{0,1\}\},$$

and let

$$B(m,n,d) = \max\{t \mid \text{there is } f \in H(t,n,d) \text{ with no copy of } K_n(m)\ f\text{-free}\}.$$

In this paper we investigate the behavior of the functions $g(p,n,d)$, $g(G,d)$, and $B(m,n,d)$ in great generality.

We determine lower and upper bounds for each of those functions, and we also obtain some exact results in some small cases.

Along this paper we make an intensive use of some versions of the celebrated Konig–Hall theorem. [See, e.g., [3, pp. 50–58].)

We also use implicitly the standard inequalities concerning $\binom{n}{m}$ and $(1 - (m/n))^t$. (See, e.g., [3, p. 255].)

Motivation

The origin of the subject of set mappings can be traced back to 1930 when some problems concerning set mappings and free sets, most of them coming from topology and set theory, were treated mostly by Polish and Hungarian mathematicians.

The subject was dormant until 1958 when Erdös and Hajnal published their almost forgotten paper "On the Structure of Set Mappings" [9].

In this fundamental work, they gave many results concerning the modification of $g(p,n,d)$ to the infinite case. They also gave some initial results on $g(p,n,d)$.

In the last few years, more work has been done. The book of Erdös *et al.* [10] summarizes the known results on the infinite case. The work of Alon, Caro, and Schönheim [1, 11, 12] shows that the finite case is no less interesting.

The first part of the current paper is devoted to both generalizing and improving the results obtained so far by Alon and Caro [1] and Erdös and Hajnal [9]. In the second and third parts we present some further generalizations and variations of the original questions, both for their own interest and because they show that our methods have a wide range of applications. It is also of interest to notice here that many classical theorems and methods of graph theory such as the König–Hall theorem, the Erdös–Stone theorem, the Ramsey theorem, and the Turan theorem, as well as counting methods and the probabilistic method, have an elegant and efficient application in the frame of set mappings.

1. THE FUNCTION $g(p,n,d)$

In this section we obtain some general results concerning the function $g(p,n,d)$. Our first result is the following.

Theorem 1.1 (upper bound). Let $g = g(p,n,d)$, $p \geqslant 2n - d$, $n > d$; then for all $r \geqslant p$

$$\frac{\prod_{j=0}^{2n-d-1} (r - j)}{(r - p + 1)n!} \geqslant \prod_{j=0}^{n-d-1} (g - n - j)\,. \tag{1.1}$$

Proof. Suppose $f \in F(g,n,d)$ is a function such that no copy of K_p^n in K_g^n is f-free. Let T be the set of all ordered pairs (e,K) where K is a copy of K_r^n in K_g^n and e is an edge of K_g^n such that $e, f(e) \in E(K)$, $|e \cap f(e)| \leqslant d$. Every copy K of K_r^n appears in at least $r - p + 1$ elements of T. Indeed, suppose this is false and let K be a copy appearing in $q \leqslant r - p$ elements of T, (e_i, K), $1 \leqslant i \leqslant q$. For each $1 \leqslant i \leqslant q$, let $v_i \in e_i$. Define $V = V(K)\backslash\{v_1, \ldots, v_g\}$. Clearly, $|V| \geqslant r - q \geqslant p > n$.

Let $e \in [V]^n \subset [V(K)]^n = E(K)$. Now, by the definition of V, $f(e) \notin E(K)$ (for otherwise, $e, f(E) \in E(K)$, which implies $e = e_j$ for some $1 \leqslant j \leqslant q$, contradicting the fact that for $1 \leqslant j \leqslant q$, $e_j \notin [V]^n$). Therefore, every copy of K_p^n containing only vertices of V is f-free, contradicting the fact that there is no such copy. Hence, we conclude that $|T| \geqslant \binom{g}{r}(r - p + 1)$. On the other hand, if e is an edge of K_g^n and $(e,K) \in T$, then K contains all the vertices of e and $f(e)$. But $|e \cup f(e)| \geqslant 2n - d$, hence e appears in at most $\binom{g-2n+d}{r-2n+d}$ elements of T, and thus

$$|T| \leqslant \binom{g}{n}\binom{g - 2n + d}{r - 2n + d}\,.$$

Combining the last two inequalities, we obtain (1.1).

Remark. For given p,n,d, the best bound is obtained by taking

$$r = \max\left\{p, \left[p - 2 + \frac{p - 1}{2n - d - 1}\right]\right\}.$$

Theorem 1.2 (lower bound). For every $p \geqslant 2n - d$, $n > d$, if g is a positive integer satisfying

$$\binom{g}{p}\left(1 - \frac{\binom{p\,-\,}{n-d}}{\binom{g-n}{n-d}}\right)^{\binom{p}{n}} < 1\,, \tag{1.2}$$

then $g(p,n,d) \geqslant g$.

Proof. Let $T \subset F(g,n,d)$ be the set of all functions f such that $|f(e) \cap e| = d$ for all $e \in E(K_g^n)$. We consider T as a probability space whose elements have equal probability. The probability that a given copy K of K_p^n is f-free is

$$\left(1 - \left(\binom{p-n}{n-d}\Big/\binom{g-n}{n-d}\right)\right)^{\binom{g}{p}}.$$

Therefore, the expected number of f-free copies of K_p^n in K_g^n is just the left-hand side of (1.2), which is smaller than 1. This shows that there exists an $f \in T$ such that no copy of K_p^n in K_g^n is f-free and establishes the theorem.

Remarks.

(1) An immediate consequence of Theorems 1.1 and 1.2 is that for a given $n, d, n > d$, there exist constants $c_1 = c_1(n,d)$, $c_2 = c_2(n,d)$ such that

$$c_2(n,d)p^t/\log p \leqslant g(p,n,d)c_1(n,d)p^t, \qquad t = \frac{2n-d-1}{n-d},$$

hence $\lim_{p\to\infty}(\log g(p,n,d)/\log p) = (2n-d-1)/(n-d)$. (See, e.g., [3, p. 255].)

(2) By an easy application of a theorem of Lovász (see [7, p. 79] we can replace the factor $\binom{g}{p}$ in (1.2) by the factor $e(\binom{p}{n}\binom{g}{p-n} + 1)$, which is better than the lower bound (1.2) for $n > d \geqslant 2$ and p large. However, this improvement is not strong enough to solve the following problem.

Problem 1. Does there exist a constant $c(n,d)$ such that $g(p,n,d) = c(n,d) \cdot P^t \cdot (1 + 0(1))$, $t = (2n-d-1)/(n-d)$?

Our last result in this section is the following.

Theorem 1.3. Let $n \geqslant k \geqslant 1$, then

$$g(n+k, n, n-k) = 2n + k\,. \tag{1.3}$$

Proof. Taking $r = p$, $g = 2n + k$, $d = n - k$ in (1.1), we find that $g(n + k, n, n - k) \leq 2n + k$. Assume $|S| = 2n + k$, then, of course, $\binom{2n+k}{n} = \binom{2n+k}{n+k}$. A simple application of the theorem of Hall and König (see [8, p. 85]) shows that there is a bijection $h: [S]^n \to [S]^{n+k}$ such that $v \subset h(v)$ for all $v \in [S]^n$. It is trivial to show that there is a function $q: [S]^n \to [S]^{n-k}$ such that $q(v) \subset v$ for all $v \in [S]^n$. (if $n - k = 0$, put $q(v) = \emptyset$). Define the function $f: [S]^n \to [S]^n$ as follows:

$$f(v) = q(v) \cup \{h(v) \setminus v\}.$$

It is easy to see that no copy of K^n_{n+k} in K^n_{2n+k} is f-free.

2. THE FUNCTION $g(G, d)$

In this section we give some results concerning the function $g(G, d)$. We only outline the proofs of the results, which are straightforward generalizations of those of Section 1. Our first result is the following.

Theorem 2.1 (lower bound). Let G be a graph. $q = |E(G)|$, $p = |V(G)|$. Denote by $h(G)$ the number of copies of G in K_p. For every edge $e \in E(G)$ define $d(e) = \deg u + \deg v - 2$ where $e = (u, v)$.

If g is a positive integer satisfying

$$h(G) \cdot \binom{g}{p} \prod_{j=1}^{q} \left(1 - \frac{d(e_j)}{2g - 4}\right) < 1, \tag{2.1}$$

then $g(G, 1) \geq g$.

If g is a positive integer satisfying

$$h(G) \binom{g}{p} \prod_{j=1}^{q} \left(1 - \frac{q - d(e_j) - 1}{\binom{g-2}{2}}\right) < 1, \tag{2.2}$$

then $g(G, 0) \geq g$.

Proof. Let $T \subset F(g, 2, 1)$ be the set of all functions f such that $|f(e) \cap e| = 1$ for all $e \in E(K_g)$. Clearly, $f(e)$ has $2g - 4$ possibilities in K_g, and if we consider a given copy of G in K_g containing e, there are $d(e)$ possibilities that $f(e) \in E(G)$. Hence, the probability that a given copy of G is f-free is $\prod_{j=1}^{q}(1 - ((d(e_j))/(2g - 4)))$. Therefore, the expected number of f-free copies of G in K_g is the left-hand side of (2.1), and the result follows. The case $d = 0$ can be proved along the same line with trivial modifications.

Remark. Obviously, $h(G) = p!/|\mathrm{Aut}\, G|$ where Aut G is the group of the automorphisms of G. In order to present an upper bound, we need some more definitions. Let G be a graph so that $p = |V(G)|$, $r \geq p$. Define $G_r = G + K_{r-p}$ (where + is the join operation). Define $h(G, r)$ as the number of copies of G in K_r. Define $h_1(G, r)$ as the number of copies of G in K_r containing a fixed copy of $K_{1,2}$. Define $h_2(G, r)$ as the number of copies of G in K_r containing a fixed copy of $2K_2$.

Theorem 2.2 (upper bound). If $g = g(G, 1)$, then for all $r \geq P$, $P = |V(G)|$.

$$\binom{g}{2} \cdot \max\{h_1(G_r, g), h_2(G_r, g)\} \geq h(G_r, g)\,(r - P + 1). \tag{2.3}$$

If $g = g(G, 0)$, then for all $r \geq P$,

$$\binom{g}{2} h_2(G_r, g) \geq h(G_r, g)\,(r - P + 1), \tag{2.4}$$

Remark. It is important to notice that if H is any induced subgraph of G_r on P vertices, then $G \subset H$, and this is a crucial point in the proof.

Proof. We prove only the first part. The second part is similar. Suppose $f \in F(g, 2, 1)$ is a function such that no copy of G in K_g is f-free. Let T be the set of all ordered pairs (e, K) where K is a copy of G_r in K_g and e is an edge of K_g such that $e, f(e) \in E(K)$, $|e \cap f(e)| \leq 1$. Every copy K of G_r appears in at least $r - P + 1$ elements of T. Indeed, suppose this is false, and let K be a copy appearing in $t \leq r - P$ elements of T, (e_i, K), $1 \leq i \leq t$. For each $1 \leq i \leq t$, let $v_i \in e_i$ be a vertex, and define $V = V(K)\backslash\{v_1, \cdots, v_t\}$. Clearly, $|V| \geq r - t \geq P$. Let $e \in [V]^2 \cap E(K)$ be an edge. By the definition of $V, f(e) \notin E(K)$. Therefore (recall the remark above), every copy of G containing only edges of $[V]^2 \cap E(K)$ is f-free, a contradiction. Hence, we conclude that

$$|T| \geq h(G_r, g)\,(r - P + 1).$$

On the other hand, if e is an edge of K_g and $(e, K) \in T$, then K contains both e and $f(e)$. Clearly, e and $f(e)$ can appear in two forms, as $K_{1,2}$ (i.e., $f(e) \cap e \neq \emptyset$) or as $2K_2$ (i.e., $f(e) \cap e = \emptyset$); hence, e appears in at most max $\{h_1(G_r, g), h_2(G_r, g)\}$ elements of T, and thus $|T| \leq \binom{g}{2} \max\{h_1(G_r, g), h_2(G_r, g)\}$. Combining the last two inequalities, we obtain (2.3).

Remarks.

(1) Clearly, $h(G_r, g) = g!/(g - r)!|\mathrm{Aut}\ G_r|$. However, if in G, $\Delta(G) \leqslant P - 2$, then $|\mathrm{Aut}\ G_r| = |\mathrm{Aut}\ G|(r - P)!$, which simplifies (2.3).

(2) Theorem 2.2 is useful for small graphs. For example, taking $g = 5$, $r = 4$ in Theorem 2.2 gives the bounds $g(P_4, 0) \leqslant 4$, $g(P_4, 1) \leqslant 4$. (P_4 is the path on four vertices.) Define a function $f: E(K_4) \to E(K_4)$ as follows: $f(1,2) = (3,4)$, $f(2,3) = (1,4)$, $f(1,3) = (2,4)$; then we conclude that $g(P_4, 0) = g(P_4, 1) = 4$.

3. THE FUNCTION $B(m,n,d)$

In this section we obtain some general results concerning the function $B(m,n,d)$. We also prove some exact results, and we present an application of this function to estimate the correlation between $l(m,H)$ and $\mathrm{ex}(m,H)$ (whose definitions are given later). Our first result is the following.

Theorem 3.1 (lower bound). If t is a positive integer satisfying

$$\binom{t}{m}^n \left(1 - \frac{2(n-1)m - 2}{2(n-1)t - 2}\right)^{\binom{n}{2}m^2} < 1\,, \tag{3.1.1}$$

then $B(m,n,1) \geqslant t$.

If t is a positive integer satisfying

$$\binom{t}{m}^n \left(1 - \frac{\binom{n}{2}m^2 - 2(n-1)m + 1}{\binom{n}{2}t^2 - 2(n-1)t + 1}\right)^{\binom{n}{2}m^2} < 1 \tag{3.1.2}$$

then $B(m,n,0) \geqslant t$.

Proof. The proof is a simple modification of the proof of Theorems 1.2 and 2.1, and we omit the details.

Theorem 3.2 (upper bound).

$$\text{If } r \geqslant m \geqslant 2, \quad \text{then} \quad B(m,2,1) \leqslant \frac{r^2(r-1)}{2(r-m)+1} + 1\,. \tag{3.2.1}$$

$$\text{If } r \geqslant m \geqslant 2, \quad \text{then} \quad B(m,2,0) \leqslant \frac{r(r-1)}{(2(r-m)+1)^{1/2}} + 1. \tag{3.2.2}$$

$$\text{If } r \geqslant m \geqslant 1, n \geqslant 3, \quad \text{then} \quad B(m,n,1) \leqslant \frac{\binom{n}{2} \cdot r^3}{2(r-m)+1}. \tag{3.2.3}$$

$$\text{If } r \geqslant m \geqslant 2, \quad \text{then} \quad B(m,3,0) \leqslant \frac{1 + \left(\dfrac{12r^3(r-1)}{2(r-m)+1} + 1\right)^{1/2}}{2}. \tag{3.2.4}$$

$$\text{If } r \geqslant m \geqslant 1, n \geqslant 4, \quad \text{then} \quad B(m,n,0) \leqslant \frac{r^2\binom{n}{2}^{1/2}}{(2(r-m)+1)^{1/2}}. \tag{3.2.5}$$

Proof. We give a detailed proof of (3.2.3) as a typical case. The proof is a modification of the proof of Theorem 1.1.

Let $t = B(m, n, 1)$ and assume the condition of (3.2.3). Suppose $f \in H(t, n, 1)$ is a function such that no copy of $K_n(m)$ in $K_n(t)$ is f-free. Let T be the set of all ordered pairs (e, K) where K is a copy of $K_n(r)$ in $K_n(t)$ and e is an edge of $K_n(t)$ such that $e, f(e) \in E(K)$.

Claim. Every copy K of $K_n(r)$ appears in at least $2(r - m) + 1$ elements of T.

Indeed, suppose this false, and let K be a copy appearing in $q \leqslant 2(r - m)$ elements of T, (e_i, K), $1 \leqslant i \leqslant q$.

For each $1 \leqslant i \leqslant q$, let $v_i \in e_i$ be a vertex, but such that no more than $r - m$ vertices belong to the same color class in $K_n(r)$. Define $V = V(K)\backslash\{v_1, \cdots, v_q\}$. Clearly, $|V| \geqslant n \cdot r - q \geqslant nr - 2(r - m) = (n - 2) \cdot (r - m) + nm$, and by the definition of V, each color class of V has at least m vertices.

Let e be an edge of the graph induced by V. Then (like in Theorem 1.1) $f(e) \notin E(V)$. Therefore, every copy of $K_n(m)$ in V is f-free, a contradiction which proves our claim. So we conclude that

$$|T| \geqslant \binom{t}{r}^n (2(r - m) + 1). \tag{1}$$

On the other hand, a simple convexity argument shows that each edge e can appear in T at most

$$\binom{t}{r}^{n-3} \cdot \binom{t-1}{r-1}^3$$

times as a left member of T.

Hence, we obtain

$$\binom{n}{2} \cdot t^2 \binom{t}{r}^{n-3} \cdot \binom{t-1}{r-1}^3 \geqslant |T|. \tag{2}$$

Combining (1) and (2) and after some algebra we obtain (3.2.3). Q.E.D.

Remark. In all cases of the theorem, the set T is defined and the inequalities are obtained in two steps.

Step A. Give a lower bound for T; it is what the claim does and it works in all the different cases.

Step B. Give an upper bound for T, by counting the maximum number of times an edge can appear in T. For this step it is necessary to distinguish between the different cases.

3.1 The Function $l(m,H)$

For $m > 2$, define

$$l(m,H) = \max\{t \mid \exists f\colon E_m \to E_m, f(e) \neq e \quad \text{for} \quad t \text{ edges of } E_m \quad \text{and no copy of } H \text{ in } K_m \text{ is } f\text{-free}\}.$$

The Turan numbers $\mathrm{ex}(m,H) = \max\{t \mid \exists\, G, e(G) = t, |G| = m, H \not\subset G\}$.

We use Theorem 3.2 together with a theorem of Erdös–Stone to strengthen Theorem 5.4 of [1].

Theorem 3.3. Let H be a graph. $\chi(H) = k \geqslant 3$, $|H| = P$, and put $r = P - k + 1$. There exists a constant $b_k > 0$ (depending only on k) such that

$$1 \leqslant \frac{l(m,H)}{\mathrm{ex}(m,H)} \leqslant 1 + \frac{k-1}{k-2} \cdot m^{-b_k/r^2}.$$

Proof. The left-hand side can be found in [1]. Clearly, the largest color class of H contains at most r vertices, $r = P - k + 1$. Hence, $H \subset K_k(r)$. Let $G(m,n)$ be a graph on m vertices and n edges such that $n > (m^2/2)(1 - (1/(k-1)) + \varepsilon)$, $(1/k) > \varepsilon > 0$. By an improved version of the Erdös–Stone theorem (see, e.g., [3, p. 328]), G contains a $K_k(t)$, (i) $t = (\alpha \log m)/(k \log 1/\varepsilon)$ (α is a constant depending only on k). We choose (ii) $t = C_1 \cdot r^2$, so that for every function $f\colon E(G) \to E(G)$ there exist $K_k(C_1 r^2)$ such that $f(e) \neq e$ for every edge e of $K_k(C_1 r^2)$. By Theorem 3.2 we conclude that G contains a $K_k(r)$-f-free graph, and hence an H-f-free. Eliminating ε from (i), (ii), we get the final result. Q.E.D.

Remark. Observe that instead of working with H, whose structure is unknown, we are working with $K_k(r)$ and $H \subset K_k(r)$. Moreover, the choice of $t = c_1 r^2$ is justified by Theorem 3.2, case (3.2.3).

As a simple application of Theorem 3.3, we prove the following.

Theorem 3.4. Denote by $P(m, G)$ the probability that a random function $f: E(K_m) \to E(K_m)$ contains a G-f-free. (G is a fixed graph.) Then

$$\lim_{m\to\infty} P(m, G) = 1 .$$

Proof. Let $P(m, n)$ denote the probability that a random function $f: E(K_m) \to E(K_m)$ has at least n edges e_i such that $f(e_i) \cap e_i \neq \emptyset$. We first prove that $\lim_{m\to\infty} P(m, 11m) = 0$. Indeed, the number of functions $f: E(K_m) \to E(K_m)$ with exactly k edges e_i such that $f(e_i) \cap e_i \neq \emptyset$ is given by

$$\text{(i)} \binom{\binom{m}{2}}{k}\binom{m-2}{2}^{\binom{m}{2}-k} \cdot (2m-3)^k .$$

Hence, the number of functions having at least $2e(2m - 3)$ edges e_i such that $f(e_i) \cap e_i = \emptyset$ is

$$\begin{aligned}
&\sum_{k=2e(2m-3)}^{\binom{m}{2}} \binom{\binom{m}{2}}{k}\binom{m-2}{2}^{\binom{m}{2}-k} \cdot (2m-3)^k \\
&\qquad \leqslant \binom{m}{2}^{\binom{m}{2}} \sum_{k=2e(2m-3)}^{\binom{m}{2}} \left(\frac{e(2m-3)}{k}\right)^k \\
&\qquad \leqslant \binom{m}{2}^{\binom{m}{2}} 2^{1-2e(2m-3)} \leqslant \binom{m}{2}^{\binom{m}{2}} 2^{-10m} \qquad (m \geqslant 20) .
\end{aligned}$$

Now $2e(2m - 3) \leqslant 11m$; hence, for $m \geqslant 20$, $P(m, 11m) \leqslant 2^{-10m}$, which is stronger than our original claim. The probability that a random function has at least $\binom{m}{2} - 11m$ edges e_i, $f(e_i) \cap e_i = \emptyset$, is therefore at least $1 - 2^{-10m}$ ($m \geqslant 20$).

Now Theorem 3.3 implies that for $m \geqslant m_0$, $\binom{m}{2} - 11m > l(m, G)$; hence, f has a G-f-free, and we proved that for $m \geqslant m_0$, $P(m, G) \geqslant 1 - P(m, 11m) \to 1$. Q.E.D.

4. VARIATIONS

In this final section, we consider only two major variations of the main subject. Undoubtedly, one can find many other interesting variations which we omitted here.

4.1 The Power Set Variation

Let $P(S)$ denote the power set of the set S, and let $P^{\alpha}(S) = P(S) \backslash \varnothing$.

Let $N(k)$ be the smallest integer N, so that for every set S with at least N elements and every function f: $P^{\alpha}(S) \rightarrow P^{\alpha}(S)$ satisfying $f(A) \not\subset A$ whenever $|A| \leqslant k$, there exists $B \subset S$ so that $|B| = k$ and $f(D) \notin P^{\alpha}(B)$ for every $D \in P^{\alpha}(B)$.

Theorem 4.1.

$$N(k) \leqslant k \cdot 2^{k-1} + k\,.$$

Proof. Let T be the set of ordered pairs (A,B) such that $|B| = k$, $B \subset S$, $A \cup f(A) \subset B$. (We assume $|S| = N = N(k) - 1$.) Clearly, $|T| \geqslant |\{B \subset S;\ |B| = k\}| = \binom{N}{k}$. On the other hand, every $A \subset B$, $|A| = j$ appears as a left element in at most $\binom{N-j-1}{k-j-1}$ elements of T. Hence, we must have

$$\sum_{j=1}^{k-1} \binom{N}{j} \binom{N-j-1}{k-j-1} \geqslant |T| \geqslant \binom{N}{k}\,.$$

Now

$$\sum_{j=1}^{k-1} \binom{N}{j} \binom{N-j-1}{k-j-1} < \sum_{j=0}^{k-1} \frac{N}{N-j} \binom{N-1}{j} \binom{N-j-1}{k-j-1}$$
$$< \frac{N}{N-k+1} 2^{k-1} \cdot \binom{N-1}{k-1},$$

and the result follows. (See [6, p. 63]).

4.2 The Arithmetic Progression Variation

Denote by $A(n,k)$ the number of arithmetic progressions $(A.P)$ of k integers in the interval $[1,n] = N_n$.

Denote by $A(k)$ the smallest integer t such that if $f{:}N_m \rightarrow N_m$, $m \geqslant t$ satisfies for every $x \in N_m$, $f(x) \neq x$; then there exists an $A.P$ of k integers $B = (a_1, \ldots, a_k)$ such that for every $a_i \in B$ $f(a_i) \notin B$.

Theorem 4.2. There exist positive constants c_1 and c_2 such that

$$c_1 k^2/\log k \leqslant A(k) \leqslant 2k^{2-c_2/\log\log k}\,.$$

Proof. We begin with the lower bound. Clearly, we have

$$A(n,k) = \sum_{j-1}^{[n/k-1]} n - j(k-1)$$

$$= n \cdot \left[\frac{n}{k-1}\right] - \frac{(k-1)\left(\left[\frac{n}{k-1}\right] + 1\right) \cdot \left[\frac{n}{k-1}\right]}{2}.$$

The probability that a given $A.P$ of length k is f-free (in the obvious meaning) is exactly $(1 - ((k-1)/(n-1))^k$. Hence, the expectation of the f-free $A.P$ of length k is $A(n,k)(1 - ((k-1)/(n-1))^k$, which is smaller than one for suitable constant $C_1 > 0$ and $n = C_1 k^2/\log k$, which proves the lower bound.

We now prove the upper bound.

Assume, as usual, that $n = A(k) - 1$, and let T be the set of all ordered pairs (v, A) where A is an $A.P$ of length k such that $v \in A$ and $f(v) \in A$. Certainly, we must have

$$|T| \geqslant A(n,k) \geqslant \frac{(n - 2k + 2) \cdot n}{2(k-1)}. \tag{i}$$

On the other hand, v and $f(v)$ can appear at most in $\binom{K}{2}$ $A.P$ of length k, according to the number of possibilities to choose (i,j) $1 \leqslant i \leqslant j \leqslant k$ such that $v = a_i$, $f(v) = a_j$. (v and $f(v)$ are given, and (i,j) are just their position in the $A.P.$) Hence, $n\binom{k}{2} \geqslant |T|$ (ii). Combining (i) and (ii) we find that

$$n \leqslant k^3 - 2k^2 + 3k - 2 < k^3.$$

However, we notice that for every choice of (i,j) such that $v = a_i$, $f(v) = a_j$, we must have

$$|i-j| \,\Big|\, |a_i - a_j| < k^3.$$

Namely, $|i-j|$ is a divisor of $|a_i - a_j|$, which depends on v only.

A well-known result in number theory [2, p. 294] states: $d(n)$, the number of divisor of n, satisfies $d(n) \leqslant n^{c_2/\log\log n}$ for a suitable constant c_2 ($c_2 = \log 2 + \varepsilon$ permitted). Since $|i-j|$ can be obtained at most $k-1$ times ($1 \leqslant i < j \leqslant k$), we infer that each integer v can appear in at most $n \cdot (k-1)k^{c_2/\log\log k}$ elements of T. Hence, after some algebra we find $n \leqslant 2k^{2+c_2/\log\log k}$. Q.E.D.

Remark. It is easy to see that $A(3) \geqslant 10$, which is perhaps exact.

We conclude this section with the following.

Conjecture. $A(k) = Ck^2 + 0(k)$.

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